On the Multi-Instanton Measure for Super Yang–Mills Theories

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Abstract

In this paper we revisit the arguments that have led to the proposal of a multi-instanton measure for supersymmetric Yang-Mills theories. We then recall how the moduli space of gauge connections on $\mathbb{R}^4$ can be built from a hyperkähler quotient construction which we generalize to supermanifolds. The measure we are looking for is given by the supermetric of the supermoduli space thus introduced. To elucidate the construction we carry out explicit computations in the case of $N = 2$ supersymmetric Yang-Mills theories.

1 Introduction

Great progresses have been made in recent years in the understanding of multi-instanton calculus. A first impulse to try to perform computations for winding numbers, k, bigger than one came from the solution for the holomorphic part of the effective action for extended globally $N = 2$ supersymmetric Yang-Mills (SYM) theories proposed in ref.[1]. The idea of checking this solution triggered a first set of computations up to $k = 2$ [2, 3, 4] in the case with no matter. Computations for $k > 2$ seemed to be out of reach due to the lack of an explicit parametrization for the ADHM data [5] and for the complexity of the algebra involved in the computation. Trying to circumvent this limitation, in ref.[6] a new computational strategy was devised: the ADHM constraints were inserted in the path integral through the introduction of a certain number of Dirac deltas. This removed the need of solving complicated algebraic equations of higher order. The product of the Dirac deltas times the differentials of the fermionic and bosonic zero–modes involved in the computation is the supermeasure that needs to be specified to perform the computation. The Dirac deltas needed to implement the constraints were found working backwards i.e. starting from the explicit form of the measure, known in the $k = 2$ case [7]. This approach was then extended to arbitrary k. The measure thus obtained possesses a certain number of desired features which are dictated by physical considerations: it is supersymmetric, reproduces the known $k = 2$ measure for $N = 1, 2$ and in the dilute gas limit factorizes as expected. The extension of this procedure to the $N = 4$ case was also given [8].

Although this procedure may seem to be rather *ad hoc*, it has anyway proved to be very useful in giving a nonperturbative consistency check of the conjectured duality [9] between certain IIB string theory correlators on an $AdS_5 \times S^5$ background and some Green's functions of composite operators of the $N = 4$ $SU(N_c)$ SYM theory in four dimensions in the large N_c limit. At leading order in that limit the $N = 4$ measure, for moduli spaces of dimension k, collapses to the product of the measure of an $AdS_5 \times S^5$ space times the partition function of an $N = 1$ ten dimensional $SU(k)$ theory reduced to 0+0 dimensions [10]. It was then possible to carry out an explicit calculation which turned out to be in agreement with the above mentioned conjecture [10]: a highly nontrivial result, in our

opinion.

Another motivation for the present study comes from ref.[11], where multi–instanton calculus was reformulated in the language of topological field theories. This gives a new geometrical interpretation of the nonperturbative effects and from a computational point of view allows to rewrite all the correlators of interest as total derivatives on the moduli space of gauge connections. This, in turn, could leads to further progresses in the computations for generic values of k's, using the properties of the ADHM contruction at the boundary of the moduli space. A key ingredient in this approach is given by the introduction of a derivative on the moduli space of gauge connections: the nilpotent BRST operator of the theory, s. If the moduli space is realized via the ADHM construction, the nilpotency of s requires the introduction of a connection which can be explicitly computed imposing the fermionic constraint of the ADHM construction [11] or, alternatively, from the Killing vectors of the residual symmetry of the above mentioned constraints. Furthermore, the BRST equations relate bosonic differentials with fermionic variables through the above cited connection, which consequently appears in the multi–instanton measure, as computed in ref.[11], by this change of basis. In view of the potential applications of these results, the scope of this paper is to revisit the derivation of the supersymmetric instanton measure in the light of the geometry of the moduli space of gauge connections. By generalizing the standard bosonic construction for the metric of a hyperkähler quotient to its supersymmetric extension, we will derive the measure from this newly introduced supermetric. The construction is valid for any k, though it can be made explicit only for $k = 2$. For higher k's we will introduce Dirac deltas, following ref.[6], which will directly implement the necessary steps of the quotient construction. This will allow us to put the derivation of the measure on a firmer mathematical basis than before since the implemented constraints do not have their origin only in symmetry arguments but stem from the quotient construction.

The plan of this paper is the following: in the next section after introducing the quotient construction, we give a brief presentation of the ADHM construction in this language and recall how to compute the bosonic metric of the moduli space in the $k = 2$

case. In the third section we recall some material from ref.[11] we need here. In the fourth section we give the supersymmetric extension of the quotient construction. Finally in the fifth and last section we show how to implement the construction of section 4 in the functional integral giving the nonperturbative contribution for arbitrary winding number k.

2 The Hyperkähler Quotient Construction

We start by discussing the Marsden-Weinstein reduction [12] which allows us to define a metric $\tilde{g}$ on the quotient $M = V/G$, where G is a Lie group with Lie algebra $\mathfrak{g}$ which acts by isometries on a Riemannian manifold (V, g). An alternative discussion of this construction, using nonlinear sigma models, can be found in ref.[13]. An application to connections on gravitational instantons is given in ref.[14]. Let us assume that the action of G is proper and free. Then the quotient $M = V/G$ is a smooth manifold, and the projection $\pi: V \to M$ is a principal bundle with structure group G. For every $\xi \in \mathfrak{g}$ the associated infinitesimal generator ξ^* of the action of G on V is a vertical fundamental vector field for the principal bundle V.

For every $x \in V$ the collection $\{\xi^*(x)\}_{\xi \in \mathfrak{g}}$ coincides with the vertical tangent space $\mathrm{Vert}_x V$. If we set $\mathrm{Hor}_x V = (\mathrm{Vert}_x V)^\perp$ in the metric g, the assignment $x \mapsto \mathrm{Hor}_x V$ is G-equivariant and therefore defines a connection on V. We shall denote by C the corresponding connection form (as a $\mathfrak{g}$-valued form on V). We have now an associated *horizontal lift operator:* for every vector field α on M, its horizontal lift $\tilde{\alpha}$ is the unique G-invariant vector field on V which projects to α.

The metric g induces a metric $\tilde{g}$ on M, given by

$$\tilde{g}(\alpha, \beta) = g(\tilde{\alpha}, \tilde{\beta}). \tag{2.1}$$

Let us write $\tilde{g}$ in components. Given local coordinates $(y^1, \ldots, y^m)$ in M and a basis $\{\xi_a\}$ of $\mathfrak{g}$ we may represent the horizontal lift in the form

$$\widetilde{\frac{\partial}{\partial y^i}} = \frac{\partial}{\partial y^i} - C_i^a \, \xi_a^*. \tag{2.2}$$

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Given the definition (2.2) for the horizontal lift and keeping in account that $\{\xi^*(x)\}_{\xi \in \mathfrak{g}}$ coincides with the vertical tangent space, the following identity holds

$$0 = g\left(\widetilde{\frac{\partial}{\partial y^i}}, \xi_a^*\right) = g\left(\frac{\partial}{\partial y^i}, \xi_a^*\right) - C_i^b \, g(\xi_b^*, \xi_a^*). \tag{2.3}$$

The matrix $g_{ab} = g(\xi_b^*, \xi_a^*)$ is invertible; by denoting by g^{ab} the elements of the inverse matrix, we get

$$C_i^a = g^{ab} \, g\left(\frac{\partial}{\partial y^i}, \xi_b^*\right). \tag{2.4}$$

Acting with the metric g on two elements of the horizontal lift (2.2), we get

$$\tilde{g}_{ij} = g_{ij} - g^{ab} g\left(\frac{\partial}{\partial y^i}, \xi_a^*\right) g\left(\frac{\partial}{\partial y^j}, \xi_b^*\right) = g_{ij} - C_i^a C_{aj}. \tag{2.5}$$

It is now possible to define a hyperkähler quotient in the following way: let X be a hyperkähler manifold of real dimension $4n$, with hyperkähler metric g and basic complex structures J_i, $i = 1, 2, 3$. Let ω_i be the corresponding Kähler forms. Assume that a Lie group G acts on X freely and properly by hyperkähler isometries, so that

$$\mathcal{L}_\xi \omega_i = 0, \tag{2.6}$$

for all $\xi \in \mathfrak{g}$ (here $\mathcal{L}$ is the Lie derivative). As a result, provided that $H^1(X, \mathbb{R}) = 0$, and having fixed a basis $\{\xi_a\}$ of $\mathfrak{g}$, one gets $3r$ "first integrals" f_i^a ($r = \dim G$) such that

$$0 = \mathcal{L}_{\xi_a} \omega_i = df_i^a. \tag{2.7}$$

Let V be the submanifold of X defined by the equations $f_i^a = 0$, so that $\dim V = 4n - 3r$. The group G acts freely and properly on V, and one has a quotient $M = V/G$ of dimension $4(n - r)$. Every complex structure on X, compatible with g, defines a complex structure on M, and one can prove that the quotient metric $\tilde{g}$ is hyperkähler. We can now bridge this construction with the standard ADHM one.

The starting point is the ADHM matrix $\Delta = a + bx$. Due to the symmetries of the ADHM contruction (we will later come back to this point at greater length) we may choose the matrix b so that it does not contain any moduli. Then in the case of the gauge group $SU(n)$, the matrix a can be written as

$$a = \begin{pmatrix} t & s^\dagger \\ A & -B^\dagger \\ B & A^\dagger \end{pmatrix}, \tag{2.8}$$

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where A, B are $k \times k$ complex matrices and s, t are $n \times k$ and $k \times n$ dimensional matrices. Let us introduce the $4k^2 + 4kn$–dimensional hyperkähler manifold $M = \{A, B, s, t\}$. Given the three complex structures J^i_{ab} where $i = 1, 2, 3$ and $a, b = 1, \ldots, \dim M$, we can build the 2–forms $\omega^i = J^i_{ab} dx^a \wedge dx^b$, where the x^a's are coordinates on M. The real forms ω^i allow one to define a $(2, 0)$ and a $(1, 1)$ form

$$
\begin{aligned}
\omega_\mathbb{C} &= \operatorname{Tr} dA \wedge dB + \operatorname{Tr} ds \wedge dt \ , \\
\omega_\mathbb{R} &= \operatorname{Tr} dA \wedge dA^\dagger + \operatorname{Tr} dB \wedge dB^\dagger + \operatorname{Tr} ds \wedge ds^\dagger - \operatorname{Tr} dt^\dagger \wedge dt \ .
\end{aligned}
\tag{2.9}
$$

The transformations

$$
\begin{aligned}
A &\rightarrow QAQ^\dagger \ , \\
B &\rightarrow QBQ^\dagger \ , \\
s &\rightarrow QsR^\dagger \ , \\
t &\rightarrow RtQ^\dagger \ ,
\end{aligned}
\tag{2.10}
$$

with $Q \in U(k), R \in U(n)$ leave $\omega_\mathbb{C}, \omega_\mathbb{R}$ invariant. Using a complex notation for the momenta defined in (2.7) $f^i_\xi = f^i_a \xi^a$, we write

$$
\begin{aligned}
f_\mathbb{C} &= [A, B] + st \ , \\
f_\mathbb{R} &= [A, A^\dagger] + [B, B^\dagger] + ss^\dagger - t^\dagger t \ .
\end{aligned}
\tag{2.11}
$$

$f^i_\xi = 0$ defines a hypersurface $\mathcal{N}^+$ in M, of dimension $k^2 + 4kn$. The moduli space of self–dual gauge connections, $\mathcal{M}^+$, is obtained by modding $\mathcal{N}^+$ by the reparametrizations defined in (2.10). It has dimension $\dim \mathcal{M}^+ = 4kn$ and, as we have already noticed, is hyperkähler. To make things even more explicit and as a guidance for future developments we now explicitly perform the $k = 2$ computation in the $SU(2)$ case [11]. In order to do this, we pause to adapt our notation to this case. In fact we find it convenient to introduce a quaternionic notation exploiting the isomorphism between $SU(2)$ and $Sp(1)$. The points, x, of the quaternionic space $\mathbb{H} \equiv \mathbb{C}^2 \equiv \mathbb{R}^4$ can be conveniently represented in the form $x = x^\mu \sigma_\mu$, with $\sigma_\mu = (i\sigma_c, \mathbb{1}_{2\times2}), c = 1, 2, 3$. The σ_c's are the usual Pauli matrices, and $\mathbb{1}_{2\times2}$ is the 2–dimensional identity matrix. The conjugate of x is $x^\dagger = x^\mu \bar{\sigma}_\mu$. A quaternion is said to be real if it is proportional to $\mathbb{1}_{2\times2}$ and imaginary if it has vanishing real part.

The prescription to find an instanton of winding number k is the following: introduce a $(k + 1) \times k$ quaternionic matrix linear in x

$$\Delta = a + bx \; , \tag{2.12}$$

where a has the generic form

$$a = \begin{pmatrix} w_1 & \cdots & w_k \\ & a' & \end{pmatrix} \; ; \tag{2.13}$$

a' is a $k \times k$ quaternionic matrix. The (anti–hermitean) gauge connection takes the form

$$A = U^\dagger dU \; , \tag{2.14}$$

where U is a $(k+1) \times 1$ matrix of quaternions providing an orthonormal frame of $\mathrm{Ker}\Delta^\dagger$, i.e.

$$\Delta^\dagger U \;\; = \;\; 0 \; , \tag{2.15}$$

$$U^\dagger U \;\; = \;\; \mathbb{1}_{2\times 2} \; . \tag{2.16}$$

The constraint (2.16) ensures that A is an element of the Lie algebra of the $SU(2)$ gauge group. The self–duality condition

$$^*F = F \tag{2.17}$$

on the field strength of the gauge connection (2.14) requires the matrix Δ to obey the constraint

$$\Delta^\dagger \Delta = (\Delta^\dagger \Delta)^T \; , \tag{2.18}$$

where the superscript T stands for transposition of the quaternionic elements of the matrix (without transposing the quaternions themselves). (2.18) in turn implies $\Delta^\dagger \Delta = f^{-1} \otimes \mathbb{1}_{2\times 2}$, where f is an invertible hermitean $k \times k$ matrix (of real numbers).

Gauge transformations are implemented in this formalism as right multiplication of U by a unitary (possibly x–dependent) quaternion. Moreover, A is invariant under reparametrizations of the ADHM data of the form:

$$\Delta \rightarrow Q\Delta R \; , \tag{2.19}$$

with $Q \in Sp(k+1), R \in GL(k, \mathbb{R})$. It is straightforward to see that (2.19) preserves the bosonic constraint (2.18). These symmetries can be used to simplify the expressions of a and b. Exploiting this fact, in the following we will choose the matrix b to be

$$b = -\begin{pmatrix} 0_{1 \times k} \\ \mathbb{1}_{k \times k} \end{pmatrix} . \tag{2.20}$$

Choosing the canonical form (2.20) for b, the bosonic constraint (2.18) becomes

$$a' = a'^{T} \;\;, \tag{2.21}$$

$$a^{\dagger}a = (a^{\dagger}a)^{T} \;\;. \tag{2.22}$$

This still allows for $O(k) \times SU(2)$ reparametrizations of the form (2.19), where now $R \in O(k)$,

$$Q = \begin{pmatrix} q & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & R^{T} & \\ 0 & & & \end{pmatrix} \;\;, \tag{2.23}$$

and $q \in SU(2)$. These transformations act nontrivially on the matrix a and leave b invariant. After imposing the constraint (2.18), the number of independent degrees of freedom contained in Δ (that is the number of independent collective coordinates that the ADHM formalism uses to describe an instanton of winding number k) is $8k+k(k-1)/2$; modding out the $O(k) \times SU(2)$ reparametrization transformations, we remain with $8k-3$ independent degrees of freedom. However (2.15) and (2.16) do not determine $U_0/|U_0|$, where U_0 is the first component of U; this adds three extra degrees of freedom, so that in conclusion we end up with a moduli space of dimension $8k$ (the instanton moduli space $\mathcal{M}^{+}$). It is easy to convince oneself that the arbitrariness in $U_0/|U_0|$ can be traded for the $SU(2)$ reparametrizations; in other words, one can forget to mod out the $SU(2)$ factor of the reparametrization group $O(k) \times SU(2)$ but fix the phase of the quaternion U_0 (setting for example $U_0 = |U_0|\mathbb{1}_{2 \times 2}$). This is what we will actually do in the following.

We now focus our attention on the zero–modes corresponding to the self–dual field introduced in (2.14). They must obey [7]

$$^{*}(D_{[\mu}\psi_{\nu]}) = D_{[\mu}\psi_{\nu]}, \quad D_{\mu}\psi_{\mu} = 0 \;\;, \tag{2.24}$$

where D is the covariant derivative in the instanton background, Eq.(2.14). The solution to (2.24) can be written as [7]

$$\psi = U^\dagger \mathcal{M} f (d\Delta^\dagger) U + U^\dagger (d\Delta) f \mathcal{M}^\dagger U \quad , \tag{2.25}$$

where $\mathcal{M}$ is a $(k+1) \times k$ matrix of quaternions which, in order for (2.24) to be satisfied must obey the constraint

$$\Delta^\dagger \mathcal{M} = (\Delta^\dagger \mathcal{M})^T \quad . \tag{2.26}$$

(2.24) tell us that the ψ zero–modes are the tangent vectors to the instanton moduli space $\mathcal{M}^+$; as it is well known, the number of independent zero–modes is $8k$ (the dimension of $\mathcal{M}^+$), and we would like to see how this is realized in the formalism of the ADHM construction. To this end, note that $\mathcal{M}$ has $k(k+1)$ quaternionic elements ($4k(k+1)$ real degrees of freedom) which are subject to the $4k(k-1)$ constraints given by (2.26). The number of independent $\mathcal{M}$'s satisfying (2.26) is thus $8k$, as desired. Notice that since the $O(k)$ symmetry is purely bosonic it only mods out the degrees of freedom in a, leaving those in $\mathcal{M}$ untouched. This fact will play an important role in the following.

If we work in the gauge where b has the canonical form (2.20), then (2.26) can be conveniently elaborated as follows. We put $\mathcal{M}$ in a form which parallels that for a in (2.13),

$$\mathcal{M} = \begin{pmatrix} \mu_1 & \cdots & \mu_k \\ & \mathcal{M}' & \end{pmatrix} \quad , \tag{2.27}$$

$\mathcal{M}'$ being a $k \times k$ quaternionic matrix. Plugging (2.13), (2.20), (2.27) into (2.26) we get

$$\mathcal{M}' = \mathcal{M}'^T \quad , \tag{2.28}$$

$$a^\dagger \mathcal{M} = (a^\dagger \mathcal{M})^T \quad . \tag{2.29}$$

When Δ is transformed according to (2.19), the $\mathcal{M}$'s must also be reparametrized so as to keep the constraint (2.26) unchanged. This implies that the $\mathcal{M}$'s must undergo the same formal reparametrization as Δ, that is

$$\mathcal{M} \rightarrow Q\mathcal{M}R \quad . \tag{2.30}$$

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Let us now consider the $k = 2$ case explicitly. The ADHM bosonic matrix reads

$$\Delta = \begin{pmatrix} w_1 & w_2 \\ x_1 - x & a_1 \\ a_1 & x_2 - x \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \\ a_3 & a_1 \\ a_1 & -a_3 \end{pmatrix} + b(x - x_0) \quad , \tag{2.31}$$

where $x_0 = (x_1 + x_2)/2$, $a_3 = (x_1 - x_2)/2$. We also need the expression of the matrix $\mathcal{M}$ which is defined in (2.26). Since this constraint is very similar to (2.18) (to get convinced of this fact just think that two solutions of (2.26) are given by $\mathcal{M}$ proportional to a and b, respectively) it is convenient to choose a form of $\mathcal{M}$ which parallels (2.31)

$$\mathcal{M} = \begin{pmatrix} \mu_1 & \mu_2 \\ \xi + \mathcal{M}_3 & \mathcal{M}_1 \\ \mathcal{M}_1 & \xi - \mathcal{M}_3 \end{pmatrix} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mathcal{M}_3 & \mathcal{M}_1 \\ \mathcal{M}_1 & -\mathcal{M}_3 \end{pmatrix} - b\xi \quad . \tag{2.32}$$

The bosonic constraint (2.18) now reads

$$\bar{w}_2 w_1 - \bar{w}_1 w_2 = 2(\bar{a}_3 a_1 - \bar{a}_1 a_3), \tag{2.33}$$

(2.33) is a set of three equations since both sides are purely imaginary. We decide to solve (2.33) with respect to a_1. A possible solution is

$$a_1 = \frac{1}{4|a_3|^2} a_3 (\bar{w}_2 w_1 - \bar{w}_1 w_2 + \Sigma) \quad , \tag{2.34}$$

where the imaginary part is fixed by (2.33) and the free real part has been called Σ. It is easy to see that

$$\Sigma = 2(\bar{a}_3 a_1 + \bar{a}_1 a_3). \tag{2.35}$$

The constraint (2.26) is

$$\bar{w}_2 \mu_1 - \bar{w}_1 \mu_2 = 2(\bar{a}_3 \mathcal{M}_1 - \bar{a}_1 \mathcal{M}_3) \quad , \tag{2.36}$$

and it is satisfied by

$$\mathcal{M}_1 = \frac{a_3}{2|a_3|^2} (2\bar{a}_1 \mathcal{M}_3 + \bar{w}_2 \mu_1 - \bar{w}_1 \mu_2) \quad . \tag{2.37}$$

As one can easily check, these are four real equations. The dimension of the tangent space to the moduli space is the right one without resorting to a quotient procedure. Let

us now introduce a 20–dimensional hyperkähler manifold $M = \{w_1, w_2, a_3, a_1, x_0\}$.[1] The parametrization (2.31) involves the combination $x_0 - x$ since the ADHM construction has a rigid translation symmetry. We then find handy to restrict the analysis to the 16–dimensional hyperkähler manifold $M\backslash\{x_0\}$ parametrized by the quaternionic coordinates

$$m^I = (w_1, w_2, a_3, a_1), \tag{2.38}$$

and endowed with a flat metric

$$ds^2 = \eta_{I\bar{J}}dm^I d\bar{m}^{\bar{J}} = |dw_1|^2 + |dw_2|^2 + |da_3|^2 + |da_1|^2 \quad , \tag{2.39}$$

which, following [13], can be also imagined to be the Lagrangian density of a suitable sigma model with target space $M\backslash\{x_0\}$. To keep the notation as simple as possible, we rename $\mathcal{M}^+\backslash\{x_0\}$ and $\mathcal{N}^+\backslash\{x_0\}$ as $\mathcal{M}^+$, $\mathcal{N}^+$, respectively.

(2.33) is invariant under the reparametrization group $O(2)$, whose action on the $k = 2$ quaternionic coordinates is

$$(w_1^\theta, w_2^\theta) = (w_1, w_2)R_\theta \quad ,$$
$$(a_3^\theta, a_1^\theta) = (a_3, a_1)R_{2\theta} \quad , \tag{2.40}$$

with

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad . \tag{2.41}$$

The construction of the reduced bosonic moduli space $\mathcal{M}^+$ proceeds now in two steps. First, given the $O(2)$ invariant solution (2.34) to the constraint (2.33), $\mathcal{N}^+$ turns out to be a 13–dimensional manifold, described by the set of coordinates (w_1, w_2, a_3, Σ). Second, we mod out the isometry group of $\mathcal{N}^+$ as discussed above. The instanton moduli space is then $\mathcal{M}^+ = \mathcal{N}^+/O(2)$, and it has dimension $\dim\mathcal{M}^+ = \dim\mathcal{N}^+ - k(k-1)/2|_{k=2} = 12$. As anticipated, the construction of the quotient space $\mathcal{M}^+$ leads to the connection (2.4) which can also be obtained by gauging a nonlinear sigma model [13]. In this case we get

$$C = \frac{1}{|k|^2}\eta_{I\bar{J}}(\bar{k}^{\bar{J}}dm^I + d\bar{m}^{\bar{J}}k^I) \quad , \tag{2.42}$$

[1]Notice that, since we are using a different parametrization of the ADHM space with respect to (2.8), the dimensions of the manifolds M and $\mathcal{N}^+$ are not those of the previous discussion. However, also the reparametrization groups are different, in such a way that the final dimension of the moduli space of self–dual gauge connections is the same, as it should be.

where $k^I \partial_I + \bar{k}^{\bar{I}} \bar{\partial}_{\bar{I}}$ is the $O(k)$ Killing vector with $|k|^2 = \eta_{I\bar{J}} k^I \bar{k}^{\bar{J}}$. The components of the $O(2)$ Killing vector on M leaving (2.34) invariant are

$$k^I = (-w_2, w_1, -2a_1, 2a_3) \quad . \tag{2.43}$$

Substituting (2.43) into (2.42), we get

$$
\begin{aligned}
C &= \frac{1}{2H}\Big(\bar{w}_1 dw_2 - \bar{w}_2 dw_1 + 2\bar{a}_3 da_1 - 2\bar{a}_1 da_3 + \\
&\quad + d\bar{w}_2 w_1 - d\bar{w}_1 w_2 + 2d\bar{a}_1 a_3 - 2d\bar{a}_3 a_1 \Big) \quad .
\end{aligned}
\tag{2.44}
$$

The metric $g_{I\bar{J}}^{\mathcal{N}^+}$ on the constrained hypersurface $\mathcal{N}^+$ is obtained plugging (2.34) into (2.39), and gets simplified if we introduce the variable

$$W = \bar{w}_2 w_1 \quad . \tag{2.45}$$

The hypersurface $\mathcal{N}^+$ is now described by the new set of coordinates (w_1, U, V, a_3, Σ), where

$$
\begin{aligned}
U &= \frac{W + \overline{W}}{2} \quad , \\
V &= \frac{W - \overline{W}}{2} \quad ,
\end{aligned}
\tag{2.46}
$$

are respectively the real and the imaginary part of W. The Jacobian factor associated to this change of variables is

$$d^4 w_1 dU d^3 V = |w_1|^4 d^4 w_1 d^4 w_2 \quad . \tag{2.47}$$

In the new variables, (2.39) reads

$$
\begin{aligned}
ds^2 &= \Big(1 + \frac{|w_2|^2}{|w_1|^2}\Big)|dw_1|^2 + \frac{dU^2}{|w_1|^2} + \frac{|dV|^2}{|w_1|^2} + \\
&\quad - \frac{dU}{|w_1|^2}(\bar{w}_2 dw_1 + d\bar{w}_1 w_2) + \frac{dV}{|w_1|^2}(\bar{w}_2 dw_1 - d\bar{w}_1 w_2) + \\
&\quad + |da_3|^2 + |da_1|^2 \quad ,
\end{aligned}
\tag{2.48}
$$

which, inserting (2.34), becomes

$$
ds^2 = \Big(1 + \frac{|w_2|^2}{|w_1|^2}\Big)|dw_1|^2 + \frac{dU^2}{|w_1|^2} + \frac{|dV|^2}{|w_1|^2} +
$$

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$$-\frac{dU}{|w_1|^2}(\bar{w}_2 dw_1 + d\bar{w}_1 w_2) + \frac{dV}{|w_1|^2}(\bar{w}_2 dw_1 - d\bar{w}_1 w_2) +$$

$$+\left(1 + \frac{|a_1|^2}{|a_3|^2}\right)|da_3|^2 + \frac{d\Sigma^2}{16|a_3|^2} + \frac{|dV|^2}{4|a_3|^2} +$$

$$-\frac{d\Sigma}{4|a_3|^2}(\bar{a}_1 da_3 + d\bar{a}_3 a_1) - \frac{dV}{2|a_3|^2}(\bar{a}_1 da_3 - d\bar{a}_3 a_1) \quad . \tag{2.49}$$

Also in this case, the r.h.s. of (2.49) can be regarded as the Lagrangian density of a zero–dimensional non–linear sigma model with target space $\mathcal{N}^+$. In real coordinates

$$m^A = (w_1^\mu, U, V^i, a_3^\mu, \Sigma), \tag{2.50}$$

the $O(2)$ Killing vector on this manifold has components

$$k^A = \left(-w_2^\mu, |w_1|^2 - |w_2|^2, 0, -2a_1^\mu, 8(|a_3|^2 - |a_1|^2)\right) \quad . \tag{2.51}$$

The global $O(2)$ symmetry can be promoted to a local one by introducing the connection (2.42), which on $\mathcal{N}^+$ is written as

$$\begin{aligned} C &= \frac{g_{AB}^{\mathcal{N}^+} k^B}{H} dm^A = \\ &= \frac{1}{H}\left(-2w_2^\mu dw_1^\mu + dU - 4a_1^\mu da_3^\mu + \frac{d\Sigma}{2}\right) \quad , \end{aligned} \tag{2.52}$$

where the metric $g_{AB}^{\mathcal{N}^+}$ is obtained by rewriting (2.49) in the coordinates $\{m_A\}$. Writing U in terms of w_1, w_2 by means of (2.45) and (2.46), the connection (2.52) becomes

$$C = \frac{1}{H}\left(w_1^\mu dw_2^\mu - w_2^\mu dw_1^\mu - 4a_1^\mu da_3^\mu + \frac{d\Sigma}{2}\right) \quad . \tag{2.53}$$

From (2.5), or alternatively from the gauged version of the Lagrangian given from (2.49), we can read off the metric on $\mathcal{M}^+ = \mathcal{N}^+/O(2)$ written in the $\{m^A\}$ coordinates, namely [13]

$$g_{AB}^{\mathcal{M}^+} = g_{AB}^{\mathcal{N}^+} - \frac{g_{AC}^{\mathcal{N}^+} g_{BD}^{\mathcal{N}^+} k^C k^D}{g_{EF}^{\mathcal{N}^+} k^E k^F} \quad . \tag{2.54}$$

We can now gauge fix the connection by imposing $\Sigma = 0$. This is a good gauge condition provided that the hypersurface $\Sigma = 0$ is transversal to the Killing vector $i.e.$ $d\Sigma(k) = |a_3|^2 - |a_1|^2 \neq 0$ (using (2.51)).

Finally, by using translational invariance to restore the dependence on x_0, and taking into account the Jacobian factor (2.47), we write the volume form on the moduli space of

self–dual gauge connections with winding number $k = 2$ as

$$|w_1|^4 \sqrt{g_{\Sigma=0}^{\mathcal{M}^+}} d^4 w_1 d^4 w_2 d^4 a_3 d^4 x_0 = \frac{H}{|a_3|^4} \Big||a_3|^2 - |a_1|^2 \Big| d^4 w_1 d^4 w_2 d^4 a_3 d^4 x_0 \quad , \qquad (2.55)$$

which reproduces Osborn's well–known result [7].

As we shall see in section 4, in terms of the cotangent bundle to $\mathcal{M}^+$ we may construct a supermanifold whose bosonic part is given by $\mathcal{M}^+$ itself. The norm of the field ψ will define the odd part of the supermetric of the supermanifold.

The norm of the zero modes (2.25) was computed by Corrigan to yield

$$
\begin{aligned}
< \psi | \psi > &= -\int_{\mathbb{R}^4} |\psi|^2 = 2\pi^2 \mathrm{Tr}\left[\mathcal{M}^\dagger (1 + P_\infty)\mathcal{M}\right] = \eta_{I\bar{J}} M^I \bar{M}^{\bar{J}} \\
&= |\mu_1|^2 + |\mu_2|^2 + |M_3|^2 + |M_1|^2,
\end{aligned}
\qquad (2.56)
$$

neglecting the coordinate ξ which is the "partner" of x_0. We then introduce "rotated" variables [3]

$$\mu_1 = \frac{w_2 \bar{a}_1 \mu_1'}{|w_2||a_1|}, \qquad \mu_2 = \frac{w_2 \bar{a}_1 \mu_2'}{|w_2||a_1|}. \qquad (2.57)$$

Substituting (2.37) into (2.56) we get

$$
\begin{aligned}
< \psi | \psi > = \ & (1 + \frac{|w_2|^2}{4|a^3|^2})(\mu_1')_\alpha (\mu_1')_\alpha + (1 + \frac{|w_1|^2}{4|a^3|^2})(\mu_2')_\alpha (\mu_2')_\alpha + (1 + \frac{|a_1|^2}{|a_3|^2})(M_3)_\alpha (M_3)_\alpha + \\
& \frac{|w_2||a_1|}{2|a_3|^2}(\mu_1')_\alpha (M_3)_\alpha - \frac{|w_1||a_1|}{2|a_3|^2}|(\mu_2')_\alpha (M_3)_\alpha - \frac{|w_1||w_2|}{4|a_3|^2}(\mu_1')_\alpha (\mu_2')_\alpha,
\end{aligned}
\qquad (2.58)
$$

where $(M^I)_\alpha$ are the real components of the quaternion M^I. From (2.56), after restoring the ξ dependence, we easily compute

$$\sqrt{< \psi | \psi >} = \frac{H^2}{|a_3|^4}. \qquad (2.59)$$

3   Connection with Topological Field Theories

In ref.[11] it was shown how the results of instanton calculus can be more easily derived in the framework of topological field theories [15]. Here we collect some results which will be relevant to our discussion. We use the same notation as in [11] to which we refer the

reader for a detailed exposition of this material. As it is well known [16], after twisting, the Lagrangian of $N = 2$ SYM is invariant under

$$sA = \psi - Dc \ ,$$
$$s\psi = -[c, \psi] - D\phi \ ,$$
$$s\phi = -[c, \phi] \ ,$$
$$sc = -\frac{1}{2}[c, c] + \phi. \tag{3.1}$$

The distinction between the cases with a vacuum expectation value of the scalar field equal or different to zero, which was important for the discussion in [11], is of no relevance here. We will not dwell on this subject anymore, working with (3.1) which is the simplest set of equations. The BRST operator, s, defined in (3.1) is such that $s^2 = 0$ and when the set of equations in (3.1) is restricted to the solutions of the Euler–Lagrange classical equation (*the zero modes*) it gives the derivative on the space $\mathcal{M}^+$. In (2.14) and (2.25) we already described these solutions in terms of the parameters of the ADHM construction for A and ψ. The form of the other fields appearing in (3.1) is [11]

$$c = U^\dagger sU, \tag{3.2}$$
$$\phi = U^\dagger \mathcal{M}f\mathcal{M}^\dagger U + U^\dagger \mathcal{A}U. \tag{3.3}$$

Plugging (2.14) and (2.25) into (3.1) leads to the action of the operator s on the elements of the ADHM construction

$$\mathcal{M} = s\Delta + C\Delta = \mathcal{S}\Delta \ , \tag{3.4}$$
$$\mathcal{A} = s\mathcal{M}\Delta + C\mathcal{M} = \mathcal{S}\mathcal{M} \ , \tag{3.5}$$
$$s\mathcal{A} = -[C, \mathcal{A}] \ , \tag{3.6}$$
$$sC = \mathcal{A} - CC \ , \tag{3.7}$$

i.e. this is the realization of the BRST algebra on the instanton moduli space. C is the connection we have introduced in (2.4). In the $k = 2$ case, using (2.44), (3.4) can be written as

$$(\widetilde{\mathcal{M}}_{\alpha\dot\alpha})_i = \sigma^\mu_{\alpha\dot\alpha}(K_{\mu\nu})_{ij}(s\widetilde{\Delta}_\nu)_j. \tag{3.8}$$

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We can now use (3.4) to further elaborate on the results of the previous section where we constructed hyperkähler metrics. We begin by reminding that the Kähler potential for those varieties is given by the second moment of the gauge field strength distribution [17]

$$\mathcal{K} = \int_{\mathbb{R}^4} x^2 |F|^2 = \frac{1}{2}\text{Tr}\left[a^\dagger (1 + P_\infty)a\right], \qquad (3.9)$$

where $P_\infty = \lim_{x\to\infty} P = 1 - bb^\dagger$ and $P = U(x)U(x)^\dagger$ is the projector onto $Ker\Delta^\dagger$. Defining complex derivatives by $s = \partial + \bar{\partial}$ the Kähler form is given by

$$\omega_{\mathcal{M}^+} = \partial\bar{\partial}\mathcal{K} = \frac{1}{2}\text{Tr}\left[(\mathcal{S}a)^\dagger(1 + P_\infty)\mathcal{S}a\right], \qquad (3.10)$$

with $\mathcal{S} = s + C$ the covariant derivative coming from (3.4). In the $k = 2$ case, substituting (4.14) of [11] in (3.10), one recovers (2.54).

Before closing this section we remind the reader that in the topological formalism of ref.[11] the measure arises as a consequence of (3.4). Let us see how. At the semi–classical level, any correlator which is expressed as a polynomial in the fields, becomes after projection onto the zero–mode subspace, a well–defined differential form on $\mathcal{M}^+$ [15]. Symbolically

$$\langle fields \rangle = \int_{\mathcal{M}^+} \left[(fields)\ e^{-S_{\text{TYM}}}\right]_{zero-mode\ subspace} . \qquad (3.11)$$

Let us now call $\{\widehat{\Delta}_i\}$ ($\{\widehat{\mathcal{M}}_i\}$), $i = 1, \ldots, p$, where $p = 8k$, a basis of (ADHM) coordinates on $\mathcal{M}^+$ ($T_A\mathcal{M}^+$). (3.4) thus yields $\widehat{\mathcal{M}}_i = s\widehat{\Delta}_i + (\widehat{\mathcal{C}\Delta})_i$. A generic function on the zero–mode subspace will then have the expansion

$$
\begin{aligned}
g(\widehat{\Delta}, \widehat{\mathcal{M}}) &= g_0(\widehat{\Delta}) + g_{i_1}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1} + \frac{1}{2!}g_{i_1 i_2}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2} + \ldots \\
&+ \frac{1}{p!}g_{i_1 i_2 \ldots i_p}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2}\cdots\widehat{\mathcal{M}}_{i_p} ,
\end{aligned} \qquad (3.12)
$$

the coefficients of the expansion being totally antisymmetric in their indices. Now (3.4) implies that the $\widehat{\mathcal{M}}_i$'s and the $s\widehat{\Delta}_i$'s are related by a (moduli–dependent) linear transformation K_{ij}, which is completely known once the explicit expression for $\mathcal{C}$ is plugged into the $\widehat{\mathcal{M}}_i$'s:

$$\widehat{\mathcal{M}}_i = K_{ij}(\widehat{\Delta})s\widehat{\Delta}_j . \qquad (3.13)$$

It then follows that

$$
\begin{aligned}
\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2}\cdots\widehat{\mathcal{M}}_{i_p} &= K_{i_1j_1}K_{i_2j_2}\cdots K_{i_pj_p}s\widehat{\Delta}_{j_1}s\widehat{\Delta}_{j_2}\cdots s\widehat{\Delta}_{j_p} = \\
&= \epsilon_{j_1\ldots j_p}K_{i_1j_1}K_{i_2j_2}\cdots K_{i_pj_p}\ s^p\widehat{\Delta} = \\
&= \epsilon_{i_1\ldots i_p}(\det K)\ s^p\widehat{\Delta}\ ,
\end{aligned}
\tag{3.14}
$$

where $s^p\widehat{\Delta} \equiv s\widehat{\Delta}_1\cdots s\widehat{\Delta}_p$. From (3.12), (3.13) we conclude that

$$
\begin{aligned}
\int_{\mathcal{M}^+}g(\widehat{\Delta},\widehat{\mathcal{M}}) &= \frac{1}{p!}\int_{\mathcal{M}^+}g_{i_1i_2\ldots i_p}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2}\cdots\widehat{\mathcal{M}}_{i_p} = \\
&= \int_{\mathcal{M}^+}s^p\widehat{\Delta}\ |\det K|g_{12\ldots p}(\widehat{\Delta})\ .
\end{aligned}
\tag{3.15}
$$

The determinant of K naturally stands out as *the instanton integration measure for $N = 2$ SYM theories.* This important ingredient of the calculation is obtained in standard instanton calculations as a ratio of bosonic and fermionic zero–mode Jacobians, while it emerges here in a geometrical and very direct way.

4   A Supermanifold Construction

The purpose of this section is to show that the results of the previous sections can be consistently put together into a supermanifold framework. We do this into two steps. In the first subsection we recall some general notions on supermanifolds, which we use in the second subsection to do the construction itself.

4.1   Generalities on Supermanifolds

In this subsection we give the definition of a supermanifold and show that these objects can conveniently be constructed from vector bundles. Naively, a supermanifold is a manifold with both "commuting and anticommuting" coordinates. One possible mathematical formalization of this idea is provided by the so-called Berezin-Leĭtes-Kostant approach [18, 19], where one considers an ordinary differentiable manifold X and "enlarges" its structure sheaf (the sheaf of germs of C^∞ functions on X)[2] to a sheaf $\mathcal{A}$ including anticommuting generators.

---

[2]For the definition of the notion of sheaf see e.g. [20, 21].

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Let us recall a few algebraic facts. A $\mathbb{Z}_2$-graded commutative algebra Λ is an associative unital algebra Λ over the real field $\mathbb{R}$ which has a splitting $\Lambda = \Lambda_0 \oplus \Lambda_1$ such that

$$ab = (-1)^{\alpha\beta}ba \quad \text{if} \quad a \in \Lambda_\alpha, \ b \in \Lambda_\beta. \tag{4.1}$$

The field $\mathbb{R}$ is embedded into Λ by $x \mapsto x \cdot 1$. A morphism $\phi : \Lambda \to \Lambda'$ between two such algebras is an algebra morphism which is compatible with the grading, i.e., $\phi(\Lambda_\alpha) \subseteq \Lambda'_\alpha$.

An (m, n)-dimensional supermanifold is a pair $\mathfrak{X} = (X, \mathcal{A})$, where X is an m-dimensional differentiable manifold, and $\mathcal{A}$ is a sheaf of $\mathbb{Z}_2$-graded commutative algebras on X, satisfying the following requirements:

- if $\mathcal{N}$ is the nilpotent[3] subsheaf of $\mathcal{A}$, then $\mathcal{A}/\mathcal{N}$ is isomorphic to the sheaf $\mathcal{C}_X^\infty$ of C^∞ functions on X. The quotient map $\sigma : \mathcal{A} \to \mathcal{C}_X^\infty$ is often called the *body map*.

- Locally the sheaf $\mathcal{A}$ is a sheaf of exterior algebras over the smooth functions with n generators; namely, every $x \in X$ has a neighbourhood W such that there is an isomorphism $\mathcal{A}(W) \simeq C_X^\infty(W) \otimes \wedge V$, where V is an n-dimensional vector space. This isomorphism is required to be compatible with the map σ.

If $(x^1, \ldots, x^m)$ are local coordinates in W, and $(\theta^1, \ldots, \theta^n)$ is a basis of V, the collection $(x^1, \ldots, x^m, \theta^1, \ldots, \theta^n)$ is said to be a local coordinate chart for $\mathfrak{X}$. According to the second requirement above, a local section of $\mathcal{A}$ (i.e., a superfunction on $\mathfrak{X}$) has a local expression

$$
\begin{aligned}
f \ = \ & f_0(x) + \sum_{\alpha=1}^n f_\alpha(x)\, \theta^\alpha \\
& + \sum_{\substack{\alpha,\,\beta\,=\,1,\,\ldots\,n \\ \alpha\,<\,\beta}} f_{\alpha\beta}(x)\, \theta^\alpha\theta^\beta + \ldots + f_{1\ldots n}(x)\theta^1 \cdots \theta^n.
\end{aligned} \tag{4.2}
$$

This is quite evidently the physicists' superfield expansion. The map $f \mapsto f_0$ is the coordinate expression of the map $\sigma : \mathcal{A} \to \mathcal{C}_X^\infty$.

Supervector fields and differential superforms may be introduced in terms of the notion of graded derivation of a graded commutative algebra Λ. A homogeneous graded

---

[3] An element in an algebra is nilpotent if it vanishes when raised to a finite power.

derivation $D : \Lambda \to \mathbb{R}$ is a linear map satisfying a graded Leibniz rule

$$D(ab) = D(a)b + (-)^{\alpha|D|}aD(b), \tag{4.3}$$

where $a \in \Lambda_\alpha$ and $|D| = 0, 1$. D is said to be even (odd) if $|D| = 0$ ($|D| = 1$). A graded derivation is the sum of an even and an odd homogeneous graded derivation. The space of such graded derivations will be denoted by $\mathrm{Der}_\mathbb{R}\Lambda$. The sheaf of derivations, $\mathrm{Der}\mathcal{A}$, of the sheaf of superfunctions is defined by the rule $(\mathrm{Der}\mathcal{A})(W) = \mathrm{Der}_\mathbb{R}\mathcal{A}(W)$ for any open set $W \subseteq X$. This is the sheaf of sections of supervector bundle on $\mathfrak{X}$ of rank (m, n), called the tangent superbundle to $\mathfrak{X}$; its sections are the supervector fields. The sections of the dual superbundle are the differential 1-superforms and by taking graded wedge products one defines the sheaves $\Omega_\mathfrak{X}^k$ of differential k-superforms. In the local coordinate charts introduced above a k-superform is written as

$$\omega = \sum_{\substack{p+q=k \\ i_1,\ldots,i_p = 1,\ldots,m \\ \beta_1,\ldots,\beta_q = 1,\ldots,n}} \omega_{i_1\ldots i_p,\beta_1\ldots\beta_q}(x,\theta)\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge d\theta^{\beta_1} \wedge \ldots \wedge d\theta^{\beta_q}. \tag{4.4}$$

For later use, we recall that a *supermetric* on $\mathfrak{X}$ is a graded-symmetric nondegenerate pairing $T\mathfrak{X} \otimes T\mathfrak{X} \to \mathcal{A}$. In local coordinates (x, θ) a supermetric is written as

$$g = g_{ij}\, dx^i \otimes dx^j + g_{i\alpha}dx^i \otimes d\theta^\alpha + g_{\alpha i}d\theta^\alpha \otimes dx^i + g_{\alpha\beta}\, d\theta^\alpha \otimes d\theta^\beta \tag{4.5}$$

where the matrix of superfunctions g_{ij} is symmetric and $g_{\alpha\beta}$ is skew-symmetric.

We study now the relation between supermanifolds and vector bundles. Given a rank n vector bundle E on X, the pair $(X, \mathcal{A})$, where $\mathcal{A}$ is the sheaf of sections $\mathcal{C}^\infty(\wedge E)$ of the exterior algebra bundle[4] $\wedge E$, is a supermanifold. Indeed, if $(\theta^1, \ldots, \theta^n)$ is a local basis of sections of E, a section of $\mathcal{C}^\infty(\wedge E)$ has the form of (4.2) (so the sections of E are regarded as Grassmann variables and the sections of the dual bundle, E^*, are fermion fields). As a matter of fact all supermanifolds are of this kind; to extract the data corresponding to a vector bundle from a supermanifold, heuristically we may regard the odd coordinates θ^α as the basis sections which locally generate the bundle. Intrinsically, with no reference

---

[4]The exterior bundle $\wedge E = \oplus_{i=1}^n \wedge^i E$ is the direct sum of the antisymmetrized tensor product of i copies of E.

to a coordinate system, if $(X, \mathcal{A})$ is an "abstract" supermanifold, and $\mathcal{N}$ is the nilpotent subsheaf of $\mathcal{A}$, then the quotient sheaf $\mathcal{N}/\mathcal{N}^2$ is the sheaf of sections of a vector bundle E, and $\mathcal{A} \simeq \mathcal{C}^\infty(\wedge E)$. This fact is known as *Batchelor's theorem* [22].

If a supermanifold $\mathfrak{X} = (X, \mathcal{A})$ is represented by a vector bundle E, its tangent super-bundle $T\mathfrak{X} = T_0\mathfrak{X} \oplus T_1\mathfrak{X}$ is explicitly described by the isomorphisms

$$T_0\mathfrak{X} \simeq \mathcal{A} \otimes TX, \qquad T_1\mathfrak{X} \simeq \mathcal{A} \otimes \mathcal{E}^* \tag{4.6}$$

where $\mathcal{E}$ is the sheaf of sections of E. Indeed, the derivations $\partial/\partial x^i$ locally generate TX, while, in view of the relation

$$\frac{\partial}{\partial\theta^\beta}\theta^\alpha = \delta^\alpha_\beta, \tag{4.7}$$

the odd derivations $\partial/\partial\theta^\alpha$ may be regarded as local generators for $\mathcal{E}^*$ (the sheaf of sections of the dual bundle to E). This also shows that there is a map $T_1^*\mathfrak{X} \hookrightarrow \mathcal{A}$ given by $d\theta^\alpha \mapsto \theta^\alpha$.

With this explicit representation of a supermanifold, a natural way to introduce a (block-diagonal) supermetric γ on it is to assign a Riemannian metric g for the even sector, and a nondegenerate alternate two-form χ on E (notice that the matrix of the bosonic metric must be symmetric while the matrix of the fermionic metric has to be skew-symmetric); with reference to (4.6), we have

$$\gamma(u_0 + u_1, v_0 + v_1) = g(u_0, v_0) + \chi(u_1, v_1). \tag{4.8}$$

As a particular case E may be the tangent or cotangent bundle to X, in which case $m = n$. If $E = T^*X$, an isomorphism $\mathfrak{X} \simeq (X, \mathcal{C}^\infty(\wedge T^*X))$ may be locally expressed in the form $\theta^i = \eta^i$, where the θ's are odd coordinates on $\mathfrak{X}$, and $(\eta^1, \ldots, \eta^m)$ form a basis of sections of the cotangent bundle.

Suppose now that X is a hermitian manifold with hermitian metric g; considering also the associated two-form ω, we have the data to define a block-diagonal supermetric:

$$\gamma = g_{ij}\, dx^i \otimes dx^j + \omega_{\alpha\beta}\, d\theta^\alpha \otimes d\theta^\beta. \tag{4.9}$$

So, let us consider the supermanifold $\mathfrak{X} = (X, \mathcal{A})$, where $\mathcal{A} = \mathcal{C}^\infty(\wedge T^*X)$ (here T^*X is the complexified $\mathcal{C}^\infty$ cotangent bundle). Let $(\eta^1, \ldots, \eta^n)$ be linearly independent forms

of type $(1, 0)$. Then locally we have

$$g = g_{i\bar{j}} \eta^i \otimes \bar{\eta}^{\bar{j}} \tag{4.10}$$

and the matrix $g_{i\bar{j}}$ is real and skew-symmetric.

The 1-forms $(\eta^1, \ldots, \eta^n, \bar{\eta}^1, \ldots, \bar{\eta}^n)$ provide local odd coordinates for $\mathfrak{X}$, and with all these data, and with the help of the dual of (4.6), we may define a supermetric γ for $\mathfrak{X}$, by letting

$$\gamma = g_{i\bar{j}} \eta^i \otimes \bar{\eta}^{\bar{j}} + i g_{i\bar{j}} d\eta^i \otimes d\bar{\eta}^{\bar{j}}. \tag{4.11}$$

So we use the datum provided by the specification of the 1-forms η^i to "replicate" the (even) metric g in the odd sector of the supermanifold. Notice that the bosonic metric is hermitean while the fermionic one is skew-hermitean. We find now convenient to use real coordinates to compute the superdeterminant of γ, by introducing $\theta^i = \Re \eta^i, \theta^{i+n} = \Im \eta^i$. To simplify the notation we will use the same symbol g_{ij} also to denote the metric in real coordinates. Suppose now that $(x^1, \ldots, x^{2n})$ are real coordinates for X. Let K be the matrix of the components of the 1-forms

$$\theta^i = K_{ik} dx^k, \tag{4.12}$$

over the basis (dx^k). A simple computation shows that in the supercoordinates $(x^1, \ldots, x^{2n}, \theta^1, \ldots, \theta^{2n})$ one has

$$\text{Sdet}\, \gamma = (\det K)^2. \tag{4.13}$$

Assume now further that X is hyperkähler, i.e., X has a hermitian metric g with three compatible complex structures J_i which generate the quaternion algebra (that is, $\nabla J_i = 0$ where ∇ is the Levi-Civita connection of g, and $J_i J_h = -\delta_{ih} + \varepsilon_{ihk} J_k$). The same definition — mutatis mutandis — applies to a complex supermanifold $(X, \mathcal{A})$ endowed with a hermitian supermetric. Let us still consider the case where $\mathcal{A}$ is the sheaf of sections of the exterior algebra of T^*X, and fix 1-forms η^j as before. Since equations (4.6) now reads

$$T\mathfrak{X} \simeq \mathcal{A} \otimes (TX \oplus TX), \tag{4.14}$$

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the three basic complex structures of X can be lifted to $\mathfrak{X}$ (the matrices of the complex structures in the odd sector expressed on the basis $(d\eta^i, d\bar{\eta}^{\bar{j}})$ are the same as the matrices of the complex structures in the even sector expressed on the basis $(\eta^i, \bar{\eta}^{\bar{j}})$), and these are automatically compatible with the supermetric of $\mathfrak{X}$. Thus, *the supermanifold $\mathfrak{X}$ acquires a hyperkähler structure.*

4.2   Supermanifolds by (even) group quotients

We reconsider now the construction in Section 2 in the case of supermanifolds. The basic theory of the action of a super Lie group $\mathfrak{G} = (G, \mathcal{H})$ on a supermanifold $\mathfrak{X} = (X, \mathcal{A})$ was developed in [19]. This theory is quite involved because the geometry of a supermanifold is not completely encoded in the underlying topological manifold but, of course, it is also described by the structure sheaf (sheaf of superfunctions). As a result one needs to formulate the theory in purely sheaf-theoretic terms (see e.g. [21]) or using the graded Hopf algebras of global functions on the supergroup [19]. Here we shall only notice that, provided that some conditions for the existence of a good quotient are satisfied, the action of $\mathfrak{G}$ on $\mathfrak{X}$ defines a quotient supermanifold $\mathfrak{Y} = (Y, \mathcal{B})$ such that $\dim \mathfrak{Y} = (m - p, n - q)$ if $\dim \mathfrak{X} = (m, n)$ and $\dim \mathfrak{G} = (p, q)$. As a consequence of a result proved in [23], these conditions are equivalent to the fact that the induced action of the bosonic part of the group G on X yields a good quotient Y.

The case of interest to us is simpler than the general situation for two reasons. The first is that the symmetry group G is purely bosonic, so that the dimension of the quotient is $(m - r, n)$ if $\dim G = r$. This simplification is present for any number N of supersymmetries. The second fact is typical of $N = 2$; in this case the dimension of the bosonic moduli space equals the number of fermionic zero-modes, which may be interpreted as differential 1-forms on the moduli space. From the viewpoint of the mathematical description we propose here this means that the vector bundle associated with the structure sheaf of the supermoduli space is the cotangent bundle. When this happens, one can get an action of G on $\mathfrak{X}$ from an action of G on the bosonic manifold X; indeed, the latter induces by linearization an action on T^*X (i.e., on the θ's) which is then extended to an

action on $\mathcal{A}$.

The Marsden-Weinstein reduction procedure now works as follows. Let $\mathfrak{V} = (V, \mathcal{A})$ be an (s, q)-dimensional supermanifold with a supermetric g. Assume also that in some local coordinate system $(x^1, \ldots, x^s, \theta^1, \ldots, \theta^q)$ the supermetric has a block diagonal form,

$$\gamma = g_{ij} \, dx^i \otimes dx^j + g_{\alpha\beta} \, d\theta^\alpha \otimes d\theta^\beta. \tag{4.15}$$

If G has dimension r, the quotient supermanifold $\mathfrak{M} = (M, \mathcal{B})$ has dimension (m, q) with $m = s - r$. The "body" manifold M is the quotient V/G. The connection C is "purely even," in the sense that, locally, $C = C_i \, dy^i$ where $(y^1, \ldots, y^m)$ are even local coordinates on M. The quotient metric has the form

$$\tilde{\gamma} = \tilde{g}_{ij} \, dy^i \otimes dy^j + g_{\alpha\beta} \, d\theta^\alpha \otimes d\theta^\beta \tag{4.16}$$

where the even components $\tilde{g}_{ij}$ are given by equation (2.5).

The square root of the superdeterminant of $\tilde{\gamma}$ gives the supermeasure of the super-moduli space for any number N of supersymmetries, since the latter specifies the "odd" geometry of the supermoduli space. For $N = 2$ this construction can be further specialized. The supermetric, for instance, can be computed using (4.11), and the results of sections 2 and 3 can be given a simple description in terms of supermanifold theory. To do so, we need to introduce a dictionary between the general supermanifold theory and the ADHM construction of the moduli space of instantons. We consider a super-manifold $\mathfrak{M} = (\mathcal{M}^+, \mathcal{B})$ whose bosonic part $\mathcal{M}^+$ is the moduli space of section 3. The fermionic zero-modes, according to (3.4), can be interpreted as differential 1-forms on $\mathcal{M}^+$. It is therefore natural to construct the supermanifold, $\mathfrak{M}$, as explained previously in this section, in terms of the cotangent bundle $T^*\mathcal{M}^+$, i.e, $\mathcal{B} = \mathcal{C}^\infty(\wedge T^*\mathcal{M}^+)$. Thus, $\dim \mathfrak{M} = (8k, 8k)$. Since M is hyperkähler, we get a hyperkähler structure for the su-permoduli space $\mathfrak{M}$. In the $k = 2$ case (4.11) is realized using the coordinates employed in (2.58) while (4.12) is given by (3.8). The supermeasure is straightforwardly obtained by (4.13) which coincides with the one found in [11]. This shows how the supermeasure can be obtained by a superquotient construction. In the next section we will show how to implement this construction in the functional integral which computes the correlators

of interest for generic values of k. Following [17] we can also write a potential for the hyperkähler supermetric:

$$f = a^\dagger(1 + P_\infty)a + \bar{\mathcal{M}}(1 + P_\infty)\mathcal{M}. \tag{4.17}$$

Indeed, by acting on (4.17) with the operator $\partial\bar{\partial}$, where ∂ is the holomorphic exterior differential on $\mathfrak{M}$, it is easy to recover the Kähler superform associated with the supermetric $\tilde{\gamma}$.

5 The Hyperkähler Quotient Construction and Multi-Instanton Calculus

In section 3 we have obtained the explicit form (3.15) of the instanton dominated correlators by plugging the constraints (2.34), (2.35), (2.37), (3.4) into (3.11). This is possible only for winding numbers up to $k = 2$ for which an explicit solution to the contraints is known. For arbitrary winding numbers an explicit solution to (2.22) and (2.26) (and, as a consequence, to (3.4)) is missing. It can be useful though, as it was shown in [8] for the $N = 4$ case, to have an expression of the type (3.11) in which the manifold on which to perform the integration is given by the unconstrained ADHM parameters introduced in (2.13) and the constraints are introduced by suitable Dirac deltas. In the bosonic case, which we treat first as an example, this program goes into the opposite direction of the strategy that we have adopted starting from (2.39) to end with (2.54). With respect to the computations carried on in [6], we do not have to worry about the transformation properties of our measure which is built to be hyperkähler (see (4.17)) and thus s-invariant.

At first we will stick to the $k = 2$ case since, as we shall see later, the extension to arbitrary winding numbers is straightforward. In order to make a bridge between the ADHM construction and the computations performed in section 2 let us remind the reader that the connection C introduced in (3.4) is a matrix of the form

$$C = \begin{pmatrix} 0 & 0 \\ 0 & C^{12} \\ -C^{12} & 0 \end{pmatrix} \tag{5.1}$$

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Plugging (3.4) into (2.26) leads to

$$\Delta^\dagger C\Delta - (\Delta^\dagger C\Delta)^T = (\Delta^\dagger s\Delta)^T - \Delta^\dagger s\Delta \ , \tag{5.2}$$

or, more compactly, to

$$L \cdot C = -\Lambda_C \ , \tag{5.3}$$

and to an explicit expression for C^{12} which is equal to (2.52) [11]. Let us now make the sigma-model interpretation of the computations in section 2 more explicit. The starting point is the metric (2.54) which can be interpreted [13] as the target space metric (described by the coordinates (2.50)) arising from the Lagrangian

$$\begin{aligned}
L &= g_{AB}^{\mathcal{M}^+} sm^A sm^B = \left(g_{AB}^{\mathcal{N}^+} - \frac{g_{AC}^{\mathcal{N}^+} g_{BD}^{\mathcal{N}^+} k^C k^D}{g_{EF}^{\mathcal{N}^+} k^E k^F}\right) sm^A sm^B \\
&= g_{AB}^{\mathcal{N}^+} sm^A sm^B - C^{12} C_{12} = g_{AB}^{\mathcal{N}^+} \mathcal{S}m^A \mathcal{S}m^B
\end{aligned} \tag{5.4}$$

where $\mathcal{S}m^A = (sm^A + C^{12}k^A)$ is the covariant derivative on $\mathcal{M}^+$. These formulae are derived from (2.4) and (2.5) in the case where the Lie algebra $\mathfrak{g} = SO(2)$. In this particular case the metric g^{ab} is a one by one metric and $C^{12} = HC_{12}{}^5$.

Imposing the constraint (2.34), the change of variable (2.35) and the explicit form (2.53) is now straightforward. Following [6] we write

$$1 = 16|a_3|^2 \int \delta(\frac{1}{4}tr_2\sigma^a(\Delta^\dagger\Delta - (\Delta^\dagger\Delta)^T))\delta(\bar{a}_3 a_1 \bar{a}_1 a_3 - \frac{\Sigma}{2}), \tag{5.5}$$

For consistency with the other sections where we constantly used forms, to impose the constraints we can use currents [24] instead of Dirac deltas. In fact the definition of an n-current

$$\int T = \int \sum_{i_1 < \ldots < i_n} T_{i_1\ldots i_n} dx_1 \wedge \ldots \wedge dx_n, \tag{5.6}$$

encompasses also the case where $T_{i_1\ldots i_n}$ is a distribution. So from now on when we write expressions like (5.5) we shall omit the differentials. Taking these observations into account, we can write

$$\begin{aligned}
\int e^{-L} &= \int\int dC^{12}\delta(C^{12} - \frac{1}{H}\left(w_1^\mu dw_2^\mu - w_2^\mu dw_1^\mu - 4a_1^\mu da_3^\mu + \frac{d\Sigma}{2}\right) \\
&\quad 16|a_3|^4\delta(\frac{1}{4}tr_2\sigma^a(\Delta^\dagger\Delta - (\Delta^\dagger\Delta)^T))\delta(\bar{a}_3 a_1 \bar{a}_1 a_3 - \frac{\Sigma}{2})e^{-L'},
\end{aligned} \tag{5.7}$$

---

[5]The connection C^{12} appearing in (5.4) is a one-form on the space spanned by the coordinates (2.50). The coordinate a_1 in (2.52) is to be replaced by its value (2.34).

where the integration is over the set of variables (2.38),

$$L' = \mathcal{S}m^I \mathcal{S}m^I = (\mathcal{S}a)^\dagger (1 + P_\infty)\mathcal{S}a = sm^I sm^I - CCH, \tag{5.8}$$

and C is given by (2.44). This shows that the Kähler form (5.8) descends on the quotient manifold (5.4).

The extension to the case of (3.11) is now straightforward. When projected onto the zero-modes subspace of winding number k, $S_{\text{TYM}} = [S_{\text{inst}}]_k = [S_B]_k + [S_F]_k$ and

$$[S_B]_k = 4\pi^2 \text{Tr}\left[2(\bar{v}v) \sum_{l=1}^{k} |w_l|^2 + \sum_{l,p=1}^{k} (\bar{w}_l \bar{v} w_p - \bar{w}_p \bar{v} w_l)(\mathcal{A}'_b)_{lp} \right] , \tag{5.9}$$

$$[S_F]_k = 4\pi^2 \text{Tr}\left[-2\bar{v} \sum_{l=1}^{k} \mu_l \bar{\mu}_l + \sum_{l,p=1}^{k} (\bar{w}_l \bar{v} w_p - \bar{w}_p \bar{v} w_l)(\mathcal{A}'_f)_{lp} \right] , \tag{5.10}$$

$\mathcal{A}'_b$ and $\mathcal{A}'_f$ being defined as

$$\mathcal{A}' = \mathcal{A}'_b + \mathcal{A}'_f , \tag{5.11}$$

where

$$L \cdot \mathcal{A}'_b = -\Lambda_b(v) , \tag{5.12}$$

$$L \cdot \mathcal{A}'_f = -\Lambda_f. \tag{5.13}$$

The right hand sides of (5.12) and (5.13) are given by [11]

$$[\Lambda_b]_{ij}(\Omega_0) = \bar{w}_i \Omega_0 w_j - \bar{w}_j \Omega_0 w_i, \tag{5.14}$$

$$\Lambda_f = \mathcal{M}^\dagger \mathcal{M} - (\mathcal{M}^\dagger \mathcal{M})^T \tag{5.15}$$

and

$$\lim_{|x|\to\infty} \phi \equiv \lim_{|x|\to\infty} U^\dagger \mathcal{A} U = v\frac{\sigma_3}{2i} . \tag{5.16}$$

Now $[S_B]_k$ contains only the w_l variables which are unconstrained, while

$$[S_F]_k = \overline{\mathcal{M}}_i^{\dot{A}\alpha}(h_{ij})_\alpha{}^\beta (\mathcal{M}_j)_{\beta\dot{A}} = (\mathcal{S}_i^{\dot{A}\alpha}a)^\dagger (h_{ij})_\alpha{}^\beta (\mathcal{S}_j)_{\beta\dot{A}}a, \tag{5.17}$$

where $i, j = 1, \ldots, n$ and[6] $h = -h^\dagger$. (5.17) is of the same form of (5.4), and leads to (3.15) with $g_{12\ldots p} = det(h)$. The difference from the bosonic case (5.4) is given by the fact

---

[6]In the following equation we denote by $h^\dagger$ the hermitean conjugate matrix obtained *without* complex conjugating v, *i.e.* treating v as real.

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that after expanding the fermionic action, the measure arises from the change of variables from $\mathcal{S}\Delta$ to $s\Delta$, as a consequence of (4.13).

To put (5.17) in a form similar to (5.8), in addition to the constraints introduced in (5.7) we also have to insert the following deltas to take care of the fermionic constraint (2.26), the BRS relation (3.4) and the presence of a scalar field in the action

$$
\begin{aligned}
1 &= \frac{1}{16|a_3|^4} \int d^4 M_1 \delta((\Delta^\dagger \mathcal{M}) - (\Delta^\dagger \mathcal{M})^T), \\
1 &= \int d\mu_1 \delta(\mu_1 - \mathcal{S}w_1), \\
1 &= \int d\mu_2 \delta(\mu_2 - \mathcal{S}w_2), \\
1 &= \int dM_3 \delta(\mu_1 - \mathcal{S}a_3), \\
1 &= H \int d\mathcal{A}_{12} \delta(\Delta^\dagger \mathcal{A}\Delta - (\Delta^\dagger \mathcal{A}\Delta)^T).
\end{aligned}
\tag{5.18}
$$

Consequently (3.11) gets modified to

$$
\langle fields \rangle = \int_{\mathcal{M}^+} dC d\mathcal{A} \delta(C + L^{-1}\Lambda_C) \delta(\Delta^\dagger \mathcal{A}\Delta - (\Delta^\dagger \mathcal{A}\Delta)^T) \delta((\Delta^\dagger \mathcal{M}) - (\Delta^\dagger \mathcal{M})^T)
$$
$$
\delta(\mathcal{M} - \mathcal{S}\Delta) \delta(\frac{1}{4} tr_2 \sigma^a (\Delta^\dagger \Delta - (\Delta^\dagger \Delta)^T) \delta(f(C)) \left[(fields) \ e^{-S_{\mathrm{TYM}}} \right]_{zero-mode}, \tag{5.19}
$$

In the $k = 2$ case, $f(C) = C \cdot n = \bar{a}_3 a_1 \mp a_1 a_3 - \Sigma/2$ where $n^A = (0, 0, a_3/2, \Sigma)$ is a certain direction in the moduli space.

Rotations of the type (2.41) act on $f(C)$ so that

$$
1 = \triangle_f(C) \int d\theta \delta(f(C^\theta)), \tag{5.20}
$$

where

$$
\triangle_f(C) = \frac{\delta f(C)}{\delta \theta} = \left| |a_3|^2 - |a_1|^2 - \frac{1}{8} \frac{\partial \Sigma^\theta}{\partial \theta} \right|_{\theta=0} \right|, \tag{5.21}
$$

as found in [6]. $\triangle_f(C)$ is invariant under $O(2)$ rotations. The reader will recognize in (5.20) the standard Faddeev-Popov trick. Then after multiplying (5.19) by $2\pi^{-1} \int d\theta$ and expanding the forms in the usual coordinate basis, $\delta(f(C))$ can be made to disappear using (5.20).

Acknowledgements

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